

# Infinitely many new renormalization group flows between Virasoro minimal models from non-invertible symmetries

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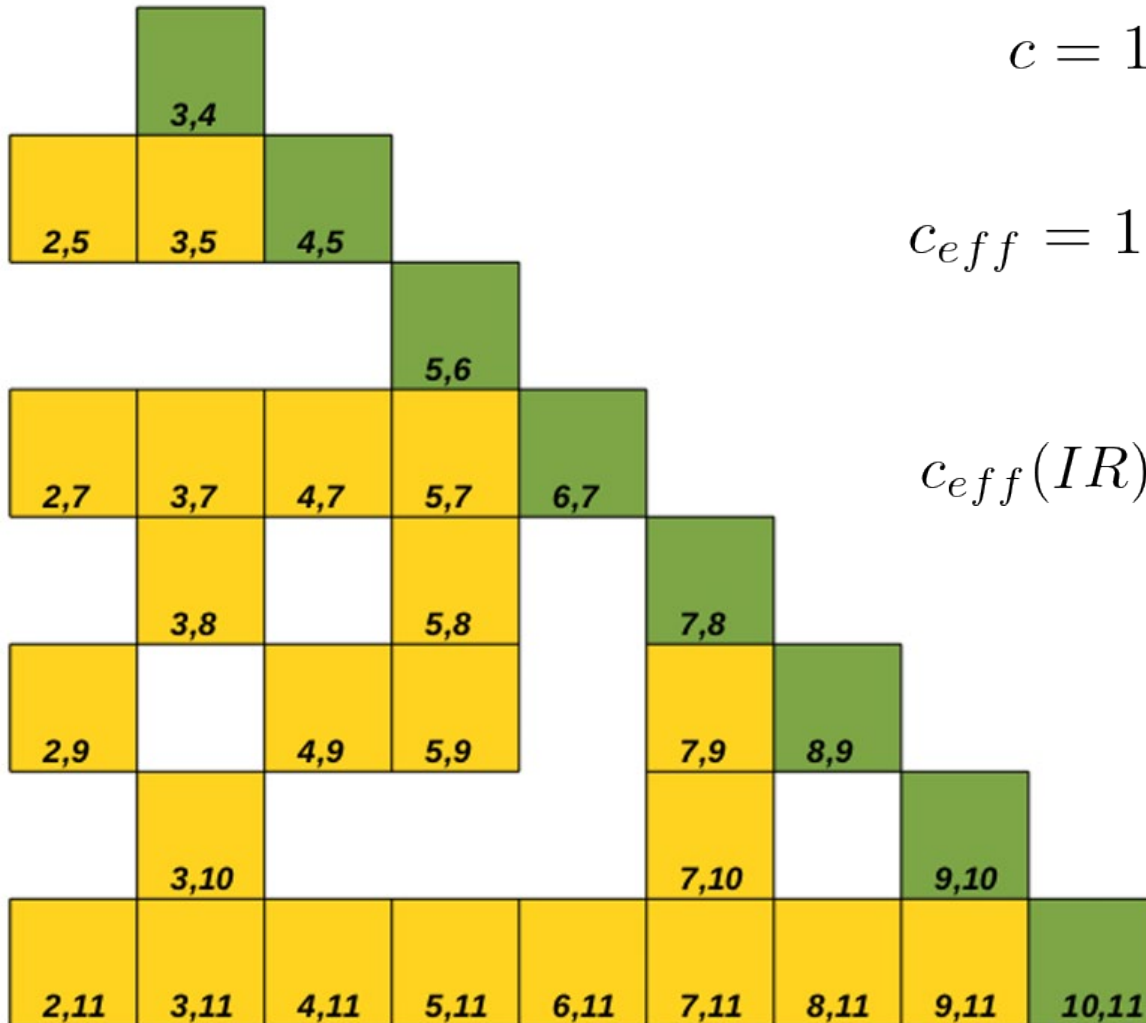
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# Virasoro minimal models

- We believe **we know everything about Virasoro minimal models**
- Specified by two coprime integers  $\mathcal{M}(p, q)$
- Central charge:  $c = 1 - \frac{6(p-q)^2}{pq}$
- When  $q = p \pm 1$ , unitary  $h_{r,s} = h_{q-r,p-s} = \frac{(pr - qs)^2 - (p-q)^2}{4pq}$ .
- Chiral spectrum from Virasoro rep theory
- Full spectrum: ADE classification
- OPE is known (A-series by Dotsenko-Fateev, D and E, Petkova)
- **But RG flows between them are poorly understood**

# RG flow

Do you know which theory is connected to which by RG flow?



$$c = 1 - \frac{6(p - q)^2}{pq}$$

$$c_{eff} = 1 - \frac{6}{pq}$$

$$c_{eff}(IR) < c_{eff}(UV)$$





$(7,11) \rightarrow (2,5)$

# Infinitely many new RG flows

$$\mathcal{M}(kq + I, q) \rightarrow \mathcal{M}(kq - I, q) \quad \text{by } \phi_{1,2k+1}$$

- This is a (infinite) generalization or **unification** of known (integrable) RG flows
  - Zamolodchikov:  $k = I = 1$
  - Ahn, Lassig:  $k = 1, I > 1$
  - Dorey, Dunning, Tateo:  $k = 2, k = 1/2$
  - Klebanov et al,  $k = 3, I = 1, q = 3$  (integrability unknown)
- Conservation of **non-invertible symmetry** is the key
- All the RG flows preserving  $SU(2)_{q-2}$  categorical symmetry are classified by our flows (no less, no more)
- All the known RG flows preserving (invertible)  $\mathbb{Z}_2$  are our flows!

Non-invertible  
symmetries

# Invertible symmetries

- Wigner claimed “symmetry” in quantum mechanics must be (projective) unitary (from conservation of probabilities)

$$U^{-1} = U^\dagger$$

- If the unitary operator commutes with Hamiltonian, it gives a **conservation of some charge**  $[U, H] = 0$   
(can be a bit more complicated if the symmetry does not commute with the Hamiltonian but shows conservation like Lorentz boost or dilatation)
- Quantum analogue of Noether’s theorem
- Useful to understand RG flows
- As we will see (most) Virasoro minimal models have (only)  $\mathbb{Z}_2$  invertible symmetry



# Non-Invertible symmetries

- Unitary ( $\rightarrow$  invertible)

$$U^{-1} = U^\dagger$$

- Commute with Hamiltonian (conservation)

$$[U, H] = 0$$

- We realize abandoning “unitary” may still give something very useful

- **Non-invertible q-form symmetry**: topological  $D - q - 1$  dimensional objects (or topological defects) in  $\text{QFT}_D$  (topological  $\rightarrow$  conservation)

- Example: topological lines that have non-trivial fusion rule

$$O_1 \times O_2 = \sum_i O_i$$

- Generically non-invertible (categorical symmetry is a better name...), and cannot be unitary

- But (as long as they are preserved) **they can be as useful to understand RG structure** as invertible symmetries

- EX: non-invertible defects in massless QED  $\rightarrow$  non-renormalization of the chiral ABJ anomaly  $\pi_0 \rightarrow 2\gamma$  (Cordova-Ohmori)

# But...

- Unless the original QFT is topological, it is not so easy to find topological defects in general QFTs

- For invertible symmetries, we can use Noether theorem to find topological defects (= conserved currents integrated over a time slice)

$$U = \exp\left(i \int d^{d-1}x j^0\right)$$

- **No systematic constructions** of general topological defects (symTFT approach?)
- For 2D CFTs one can spell out the conditions, but solving them is not so easy
- EXCEPT for A-series Virasoro minimal models where we believe **all the topological defect lines are given by Verlinde lines**

Non-invertible  
symmetries in A-series  
minimal models

# Invertible symmetries in Virasoro

minimal models  $\mathcal{M}(p, q)$   $c = 1 - \frac{6(p-q)^2}{pq}$

- Convention: Unlike yellowbook, we always fix the order of p and q

$$\mathcal{M}(5, 4) \rightarrow \mathcal{M}(3, 4) \quad h_{r,s} = h_{q-r,p-s} = \frac{(pr - qs)^2 - (p - q)^2}{4pq}$$

- Fusion rules:

$$\phi_{(r,s)} \times \phi_{(m,n)} = \sum_{\substack{k=1+|r-m| \\ k+r+m=1 \pmod{2}}}^{\min(r+m-1, 2q-1-r-m)} \sum_{\substack{l=1+|s-n| \\ l+s+n=1 \pmod{2}}}^{\min(s+n-1, 2p-1-s-n)} \phi_{(k,l)}$$

- Due to the operator identification, the symmetry of (A-series) minimal model is  $\mathbb{Z}_2$  (see e.g. Lassig)
  - (Even,Odd):  $r-1 \pmod{2}$
  - (Odd,Even):  $s-1 \pmod{2}$
  - (Odd,Odd):  $r+s-1 \pmod{2}$  (anomalous in the 't Hooft sense)
- $\mathbb{Z}_2$  of (Odd,Odd) model cannot be gauged (if it were gauged, D-series should exist, but (Odd,Odd) has only A-series modular invariant partition function)
- No RG flows between (Odd,Even) and (Odd,Odd) unless we break  $\mathbb{Z}_2$

# Verlinde lines in (A-series) minimal models

- It has one-to-one correspondence with (chiral) Virasoro character

$$h_{r,s} = h_{q-r,p-s} = \frac{(pr - qs)^2 - (p - q)^2}{4pq}.$$

- Action of topological lines on states

$$L_{(r,s)} |\phi_{(\rho,\sigma)}\rangle = \frac{S_{(r,s),(\rho,\sigma)}}{S_{(1,1),(\rho,\sigma)}} |\phi_{(\rho,\sigma)}\rangle,$$

- Explicit modular S-matrix can be found in any CFT textbook

$$S_{(r,s),(\rho,\sigma)} = 2\sqrt{\frac{2}{pq}} (-1)^{1+s\rho+r\sigma} \sin\left(\pi\frac{p}{q}r\rho\right) \sin\left(\pi\frac{q}{p}s\sigma\right),$$

- Fusion rule is same as (chiral) Virasoro fusion rule

$$\begin{aligned} L_{(r,s)} \times L_{(m,n)} &= \sum N_{(r,s)(m,n)}^{(k,l)} L_{(k,l)} \\ &= \sum_{\substack{k=1+|r-m| \\ k+r+m=1 \pmod{2}}}^{\min(r+m-1, 2q-1-r-m)} \sum_{\substack{l=1+|s-n| \\ l+s+n=1 \pmod{2}}}^{\min(s+n-1, 2p-1-s-n)} L_{(k,l)}. \end{aligned}$$

- Example: Duality topological defect lines  $\rightarrow$  Tambara-Yamagami fusion category

$$\eta \times \eta = 1 + N, \quad \eta \times N = \eta, \quad N \times N = 1$$

- The consistency (e.g. Cardy condition) is guaranteed by the Verlinde formula

$$N_{ab}^c = \sum_d \frac{S_{ad} S_{bd} S_{dc}}{S_{0d}},$$

# RG constraint from non-invertible symmetries

- Assume the deformation preserves a topological defect line

$$L_a \phi_b |\Phi\rangle = \phi_b L_a |\Phi\rangle \quad \text{on any states } |\Phi\rangle$$

(in unitary theories checking on vacuum is sufficient)

- Assume CFT1 becomes CFT2 (in our case we assume it will be another A-series Virasoro minimal model)
- What kind of properties of topological defect lines are preserved?
  - Quantum dimensions of topological defect lines
  - Spin contents of topological defect lines(It will turn out that the constraints are the same)  
See e.g. Chang-Lin-Shao-Wang-Yin

# Quantum dimensions of topological defect lines

- Defined by the action of topological defect lines on the vacuum states

$$L_{(r,s)} |0\rangle = d_{(r,s)} |0\rangle = \frac{S_{(r,s),(1,1)}}{S_{(1,1),(1,1)}} |0\rangle,$$

- Interpret it as the expectation value on the cylinder

$$d_a = \langle 0 | L_a | 0 \rangle = \langle L_a \rangle \quad a := (r, s)$$

- Satisfies the fusion constraints ( $\rightarrow$  only discrete solutions)

$$\langle L_a \rangle \langle L_b \rangle = \sum_d N_{ab}^d \langle L_d \rangle$$

- E.g. invertible  $\mathbb{Z}_2$  symmetry  $\langle L_{\mathbb{Z}_2} \rangle \langle L_{\mathbb{Z}_2} \rangle = \langle 1 \rangle$

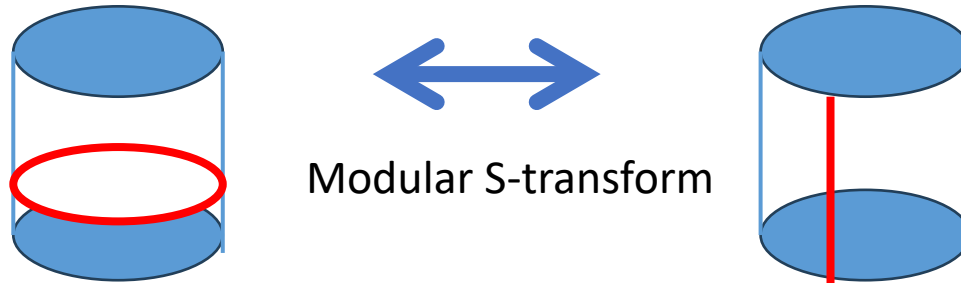
- Plus  $\rightarrow$  non-anomalous

- Minus  $\rightarrow$  anomalous

$$\langle L_{\mathbb{Z}_2} \rangle = \pm 1$$

- Cannot be changed by continuous deformations  $\rightarrow$  RG invariants (rigidity of modular tensor category)

# Spin contents of Verlinde lines



- We focus on spin contents of the defect Hilbert space

$$\begin{aligned} Z_{L_a}(\tau, \bar{\tau}) &= \sum_{b,c,d} \frac{S_{ab}}{S_{0b}} S_{bc} S_{bd} \chi_c(\tilde{\tau}) \bar{\chi}_d(\bar{\tilde{\tau}}) \\ &= \sum_{c,d} N_{cd}^a \chi_c(\tilde{\tau}) \bar{\chi}_d(\bar{\tilde{\tau}}) \end{aligned}$$

- Spin  $h_c - \bar{h}_d$  in defect Hilbert space may not be (half) integer (but OK)
- Since RG flow commute with rotation, **spin contents (mod integer) will be conserved!** (← novel RG constraints from categorical symmetry!)



New RG flows from non-invertible symmetries

$\mathcal{M}(kq + I, q) \rightarrow \mathcal{M}(kq - I, q)$  by  $\phi_{1,2k+1}$

- We can show  $\phi_{1,2k+1}$  or more generally  $\phi_{1,2l+1}$  preserves  $L_{(1,1)}, \dots, L_{(q-1,1)}$  (when  $k$  is integer)

$$L_a \phi_b |\Phi\rangle = \phi_b L_a |\Phi\rangle$$

- Proof: direct computation

$$L_{(i,1)}^{UV} = L_{(i,1)}^{IR} \quad L_{(q-1,1)} = \mathbb{Z}_2$$

- We can show that under these proposed RG flows
  - **Quantum dimensions** of  $L_{(i,1)}$  are preserved
  - **Spin contents** of  $L_{(i,1)}$  are preserved
- We can further show that the proposed flows are **sufficiently fine-grained**: each connected RG flows have different quantum dimensions/spin contents

# Quantum dimensions under our RG flows

- In  $\mathcal{M}(kq + I, q) \rightarrow \mathcal{M}(kq - I, q)$  by  $\phi_{1,2k+1}$   
 $L_{(1,1)}, \dots, L_{(q-1,1)}$  are preserved
- Check of the **matching of quantum dimensions** of  $L_{(r,1)}$

$$\frac{d_{(r,1)}^{UV}}{d_{(r,1)}^{IR}} = \frac{\sin(\pi \frac{kq+I}{q} r) \sin(\pi \frac{kq-I}{q})}{\sin(\pi \frac{kq-I}{q} r) \sin(\pi \frac{kq+I}{q})} = \frac{\sin(\pi \frac{I}{q} r) \sin(-\pi \frac{I}{q})}{\sin(-\pi \frac{I}{q} r) \sin(\pi \frac{I}{q})} = 1.$$

- They have different quantum dimensions. The most severe constraint comes from  $L_{(2,1)}$

$$d_{(2,1)} = -2 \cos\left(\frac{p}{q}\pi\right)$$

- Takes a distinct values for different  $p \pmod{q}$  with a given  $q$ . (There exist  $\varphi(q)$  different RG paths)

Examples and physical  
interpretations

$\mathcal{M}(3k + I, 3) \rightarrow \mathcal{M}(3k - I, 3)$  and anomaly matching

- Preserved topological line defect is only  $\mathbb{Z}_2$  invertible symmetry  $L_{(2,1)} = \mathbb{Z}_2$
- Our RG flows predict (Odd, 3)  $\rightarrow$  (Odd,3) and (Even,3)  $\rightarrow$  (Even,3)
- They are classified by the quantum dimensions of  $L_{(2,1)}$ 
$$d_{(2,1)} = \pm 1$$
- This is nothing but the 't Hooft anomaly matching
  - Recall (Odd,Odd) case is anomalous
- Our RG flow knows 't Hoof anomaly matching

# $\mathcal{M}(4k + I, 4) \rightarrow \mathcal{M}(4k - I, 4)$ and duality defects

- Preserved topological defect lines are  $\mathbb{Z}_2$  invertible symmetry  $L_{(3,1)}$  and non-invertible “duality defect”  $L_{(2,1)}$
- That has Tambara-Yamagami (=Ising) fusion rule  
 $\eta \times \eta = 1 + N$ ,  $\eta \times N = \eta$ ,  $N \times N = 1$ 

$$N = L_{(3,1)}$$

$$\eta = L_{(2,1)}$$
- **Only (p,4) minimal models have a duality defect**
- Suppose we gauge  $\mathbb{Z}_2$  in half space-time

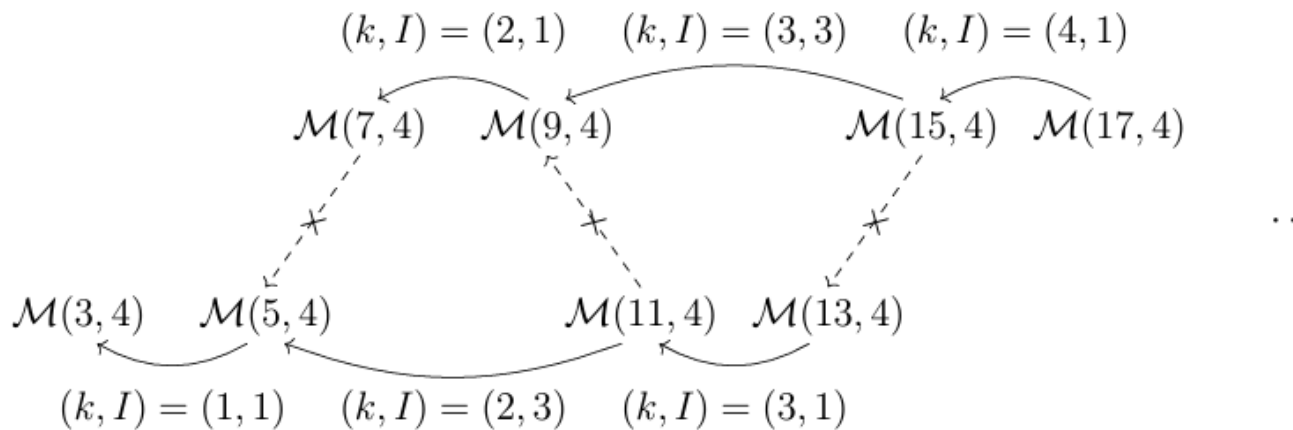


- If  $\mathbb{Z}_2$  is non-anomalous we will get D-series minimal model
- But only in (p,4), A-series and D-series are same (self-dual)
- Half gauging gives the non-trivial topological defect line  

$$\eta = L_{(2,1)}$$

# $\mathcal{M}(4k + I, 4) \rightarrow \mathcal{M}(4k - I, 4)$ and duality defects

- We have **two distinct duality defect lines**
- Quantum dimensions  $d_{(2,1)} = \pm\sqrt{2}$  distinguish  $\varphi(4) = 2$  distinct RG flows



- If you study the spectrum, the number of “singlet” relevant deformations decrease one by one along the proposed flow (but not in the forbidden flow)
- Dotted arrows can be realized in half integer  $k$  flow which breaks the duality symmetry

# Application: fate of non-SUSY Yukawa fixed point (Nakayama-Kikuchi)

- Study Yukawa theory in  $d=4-\epsilon$  (Fei-Giombi-Klebanov-Tarnopolsky)

$$S = \int d^d x (\partial_\mu \phi \partial^\mu \phi + i \bar{\psi} \gamma^\mu \partial_\mu \psi + g_1 \phi \bar{\psi} \psi + g_2 \phi^4)$$

- One (stable) fixed point is supersymmetric  $\rightarrow$  fermionic  $\mathcal{M}(5, 4)$  in  $d=2$
- The other (unstable) fixed point without SUSY  $\rightarrow$  fermionic  $\mathcal{M}(?, 4)$  in  $d=2$ ?
- Chiral symmetry = non-invertible duality defect
- Must flow to  $\mathcal{M}(5, 4)$
- Cannot be  $\mathcal{M}(7, 4)$  or  $\mathcal{M}(9, 4)$  but  $\mathcal{M}(11, 4)$ !



# Discussions and Conclusions

# Comments on integrability

- $\mathcal{M}(kq + I, q) \rightarrow \mathcal{M}(kq - I, q)$  by  $\phi_{1,2k+1}$
- $l = 1$  case seems **integrable** in the TBA sense
- $l > 1$  case: **no TBA is known**, but may be integrable in the sense of “integral equations” (e.g. Dorey-Dunning-Tateo)
- I need your help

# Massive flow and TQFTs

- $\mathcal{M}(kq + I, q) \rightarrow \mathcal{M}(kq - I, q)$  by  $\phi_{1,2k+1}$
- One sign gives massless flow (our flow)
- The other sign should give massive flow
- We may study massive excitations of S-matrix (e.g. Coppeti-Cordova-Komatsu), vacuum structure and IR TQFTs
- (Due to the non-invertible symmetry, “ground states” must be degenerate  $\rightarrow$  Non-trivial TQFTs)
- I need your help

# Puzzle in $(2, q)$ and $(p, 2)$

- $\mathcal{M}(kq + I, q) \rightarrow \mathcal{M}(kq - I, q)$  by  $\phi_{1, 2k+1}$
- We can formally put  $q=2$ : (multi-critical Lee-Yang flow)  
$$\mathcal{M}(2k + I, 2) \rightarrow \mathcal{M}(2k - I, 2)$$
- $q = 2$  case has no  $\mathbb{Z}_2$
- But  $kq - I$  can be 2
- $\mathbb{Z}_2$  preserving deformations of  $\mathbb{Z}_2$  symmetric theory gives no  $\mathbb{Z}_2$  ? SSB or decoupling?

# Summary

- Non-invertible symmetries give very powerful constraint on RG flows
- Infinitely many constraints and classifications than just invertible symmetries
- Other 2D CFTs?
- Even in unitary minimal models E-series flows are barely understood
- There should be very powerful constraints **in higher dimensions** from non-invertible symmetries (if we can find them systematically)